Vectors

For physics and calculus students

Prepared by Larry Friesen and Anne Gillis

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This project is a direct result of math/physics instructional discontinuity identified while attending the MAC^3 workshop in August, 2006

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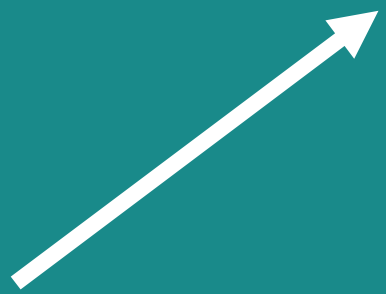
Vectors – why?

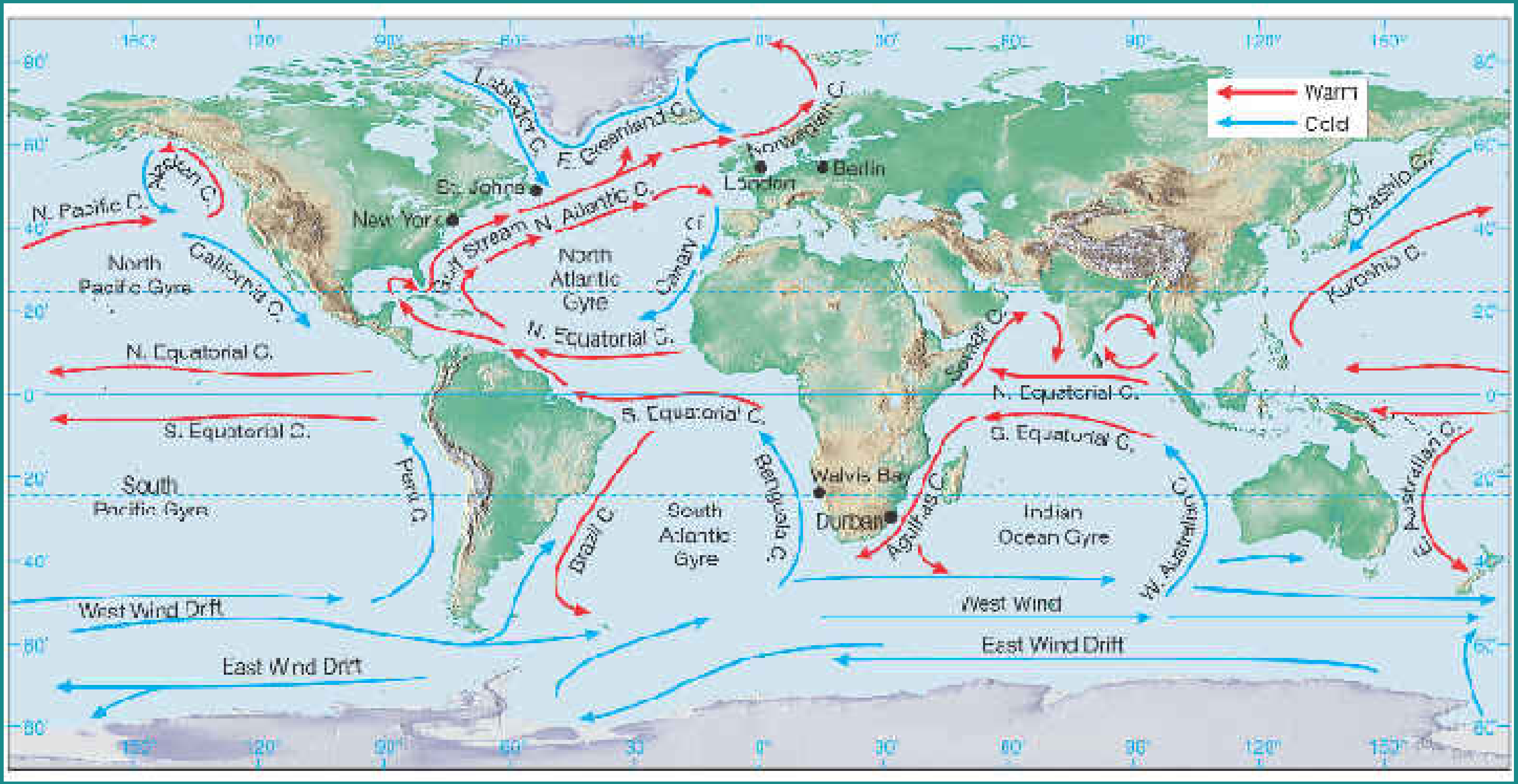
- Wherever we have a quantity with both a direction and a magnitude (amount or size), both pieces of information may be efficiently handled using vectors.

Vector (direction matters)	Non-vector = “scalar”
velocity	Energy
force	Temperature
Electric and magnetic fields	age
	numbers

Vector Representations

- There are many ways to represent a vector.
- The simplest, visual way is with an arrow.

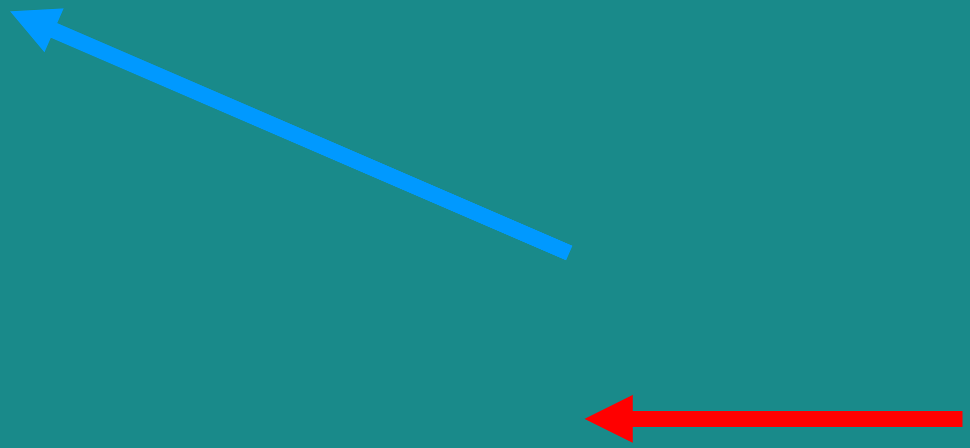




- In this map of ocean currents above, the arrows indicate the direction of the movement of water

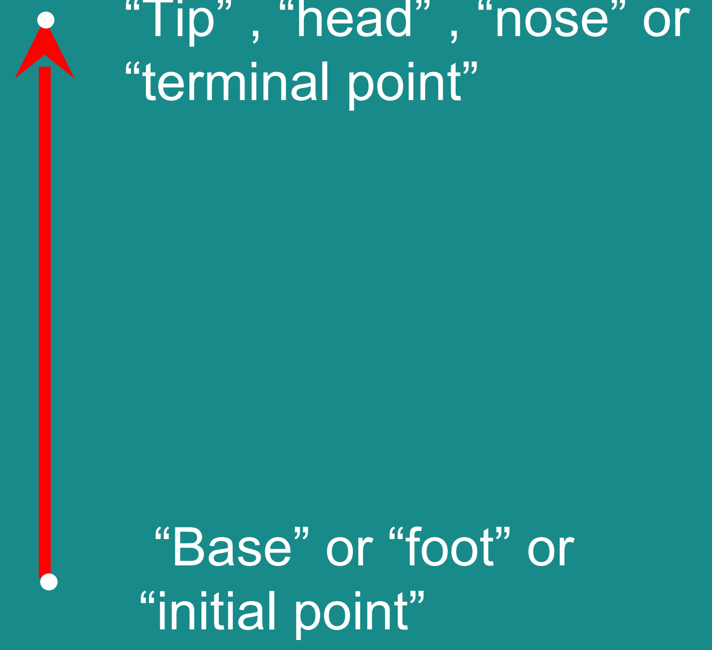
(image courtesy Hunter College, City University of New York)

- Actual vectors are drawn using straight lines, however.

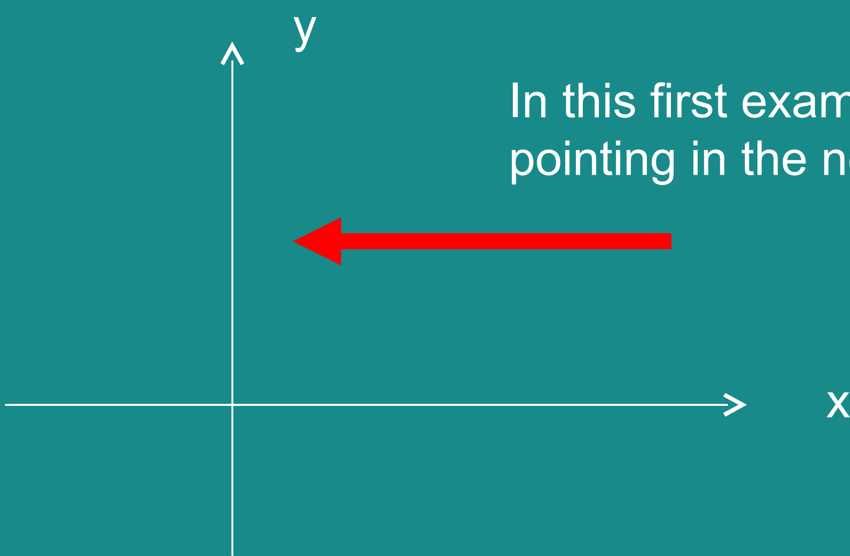


Vector Anatomy

- There are several different words used to describe the ends of the arrow

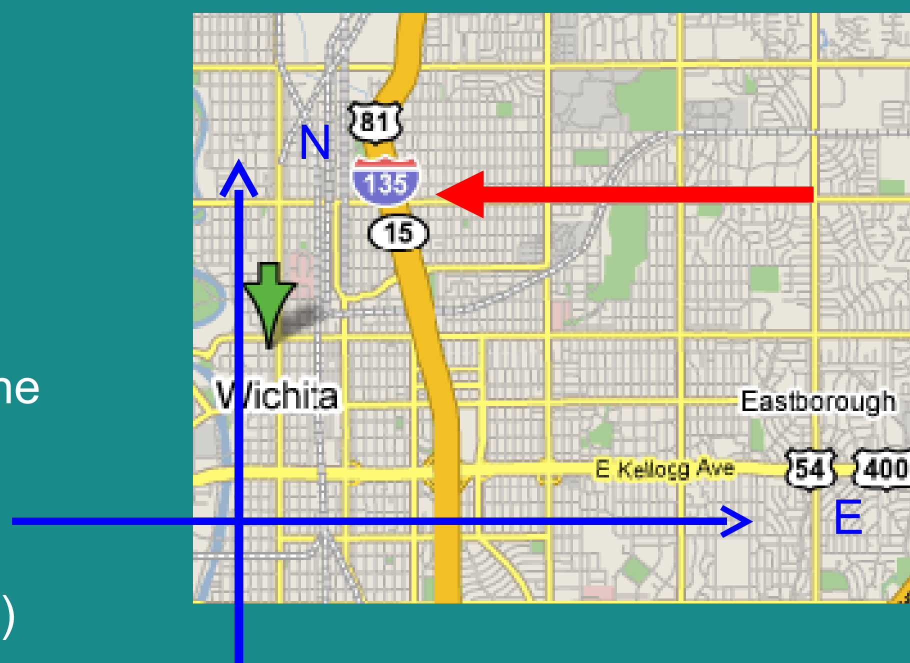


For the direction of the arrow to be meaningful, some sort of coordinate system is necessary



In this first example, the vector is pointing in the negative x direction

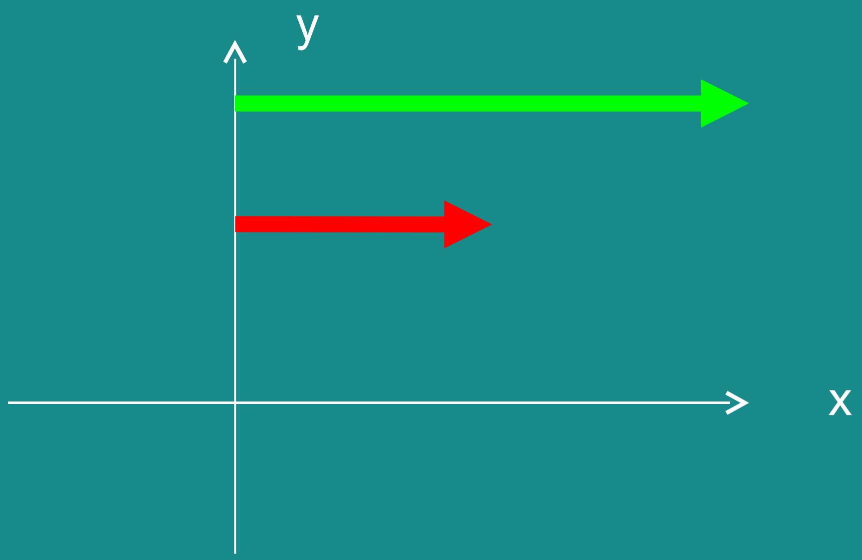
In this second example, the vector is pointing West



(Google maps)

Scale

- The length (or "magnitude") of the vector is as important as its direction. Vectors are usually drawn to scale.
- The green arrow is twice as long as the red arrow which indicates it has twice the magnitude.

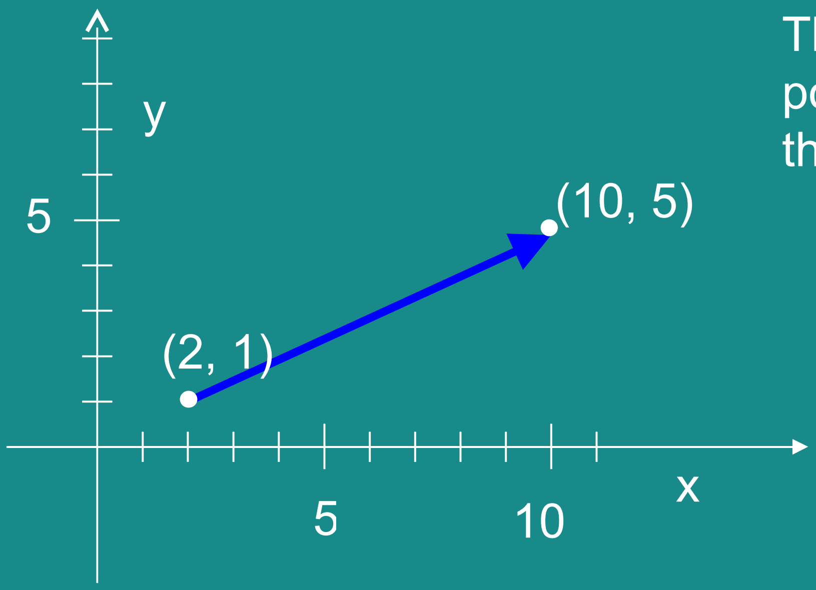


If these were velocity vectors and the green vector represented a velocity of 10 m/s in the positive x direction, then the red vector would be interpreted as 5 m/s in the positive x direction.

End points

- Typically use a Cartesian coordinate system
- In calculus, end and starting points important

The vector starts at (x_1, y_1) and ends at (x_2, y_2)

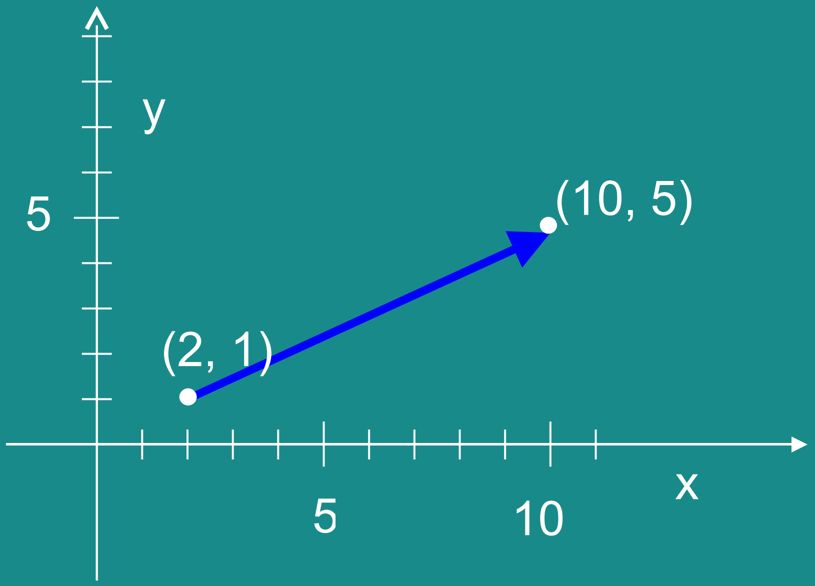


This vector starts on the point (2, 1) and ends on the point (10, 5)

Writing Vectors as an Ordered Pair

- One handy way to write a vector is as an ordered pair.
- Use the end points and calculate $< x_2-x_1 , y_2-y_1 >$

So this vector can be written <10 - 2, 5 -1> which equals <8, 4>.

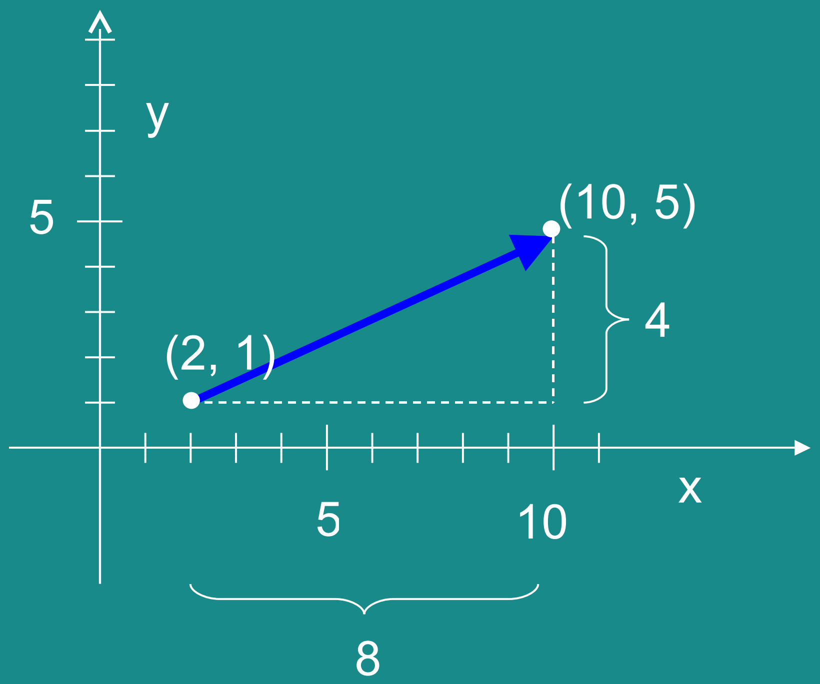


This is the vector <8,4>.

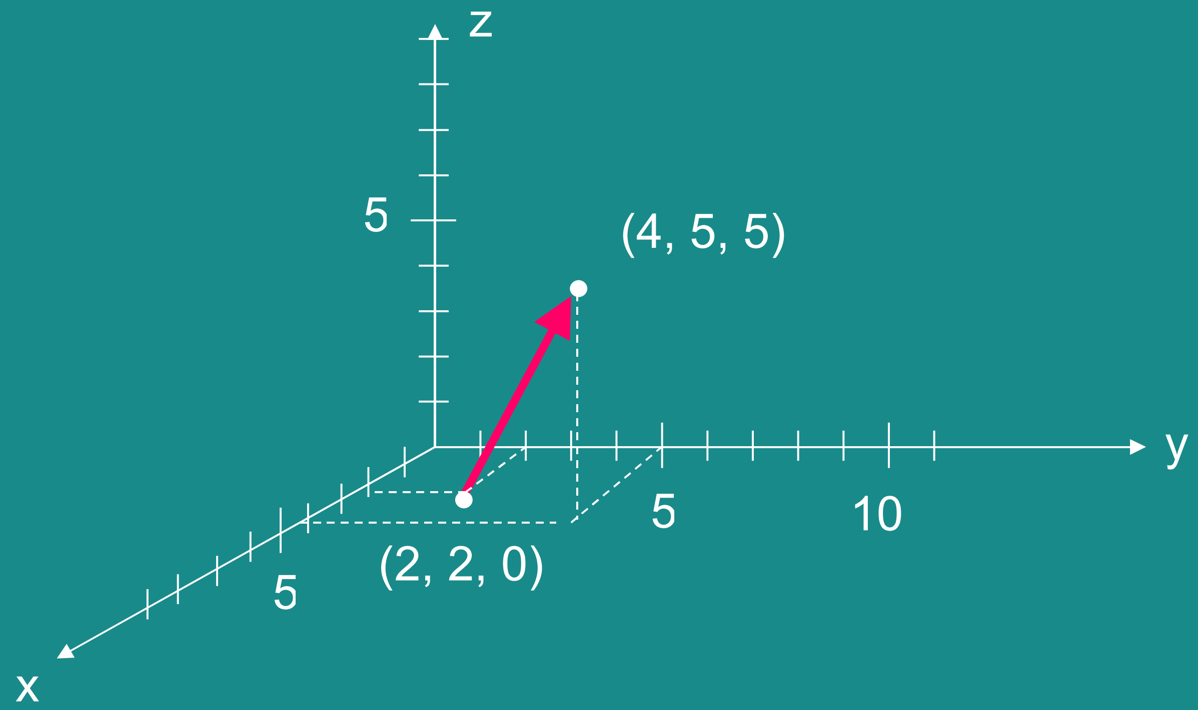
You can think of it as meaning go to the right 8 and up 4.

Components

- For the vector <8,4>, the "8" and the "4" are the x and y "components" for the vector.

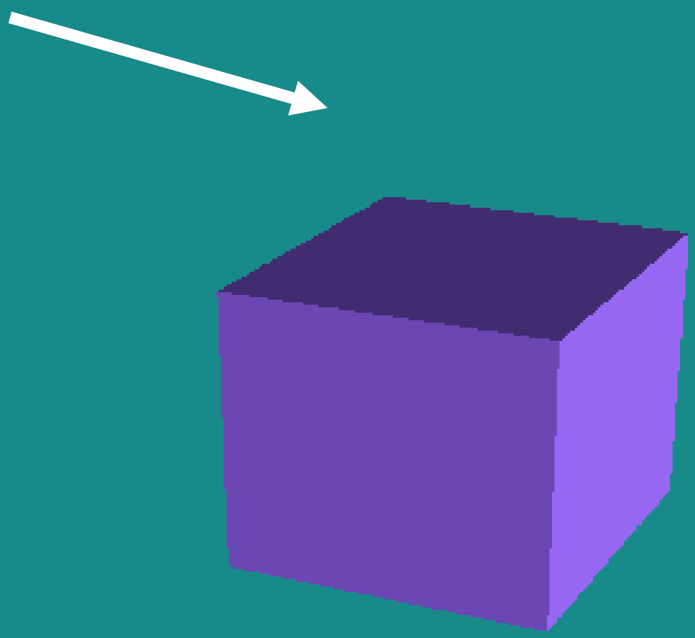


In 3D



- The subtraction is extended to three dimensions by $<x_2-x_1, y_2-y_1, z_2-z_1>$
- This is the vector <2, 3, 5>
- It's x-component is 2, it's y-component is 3 and its z-component is 5

- In physics, where the vector is located is not always important.

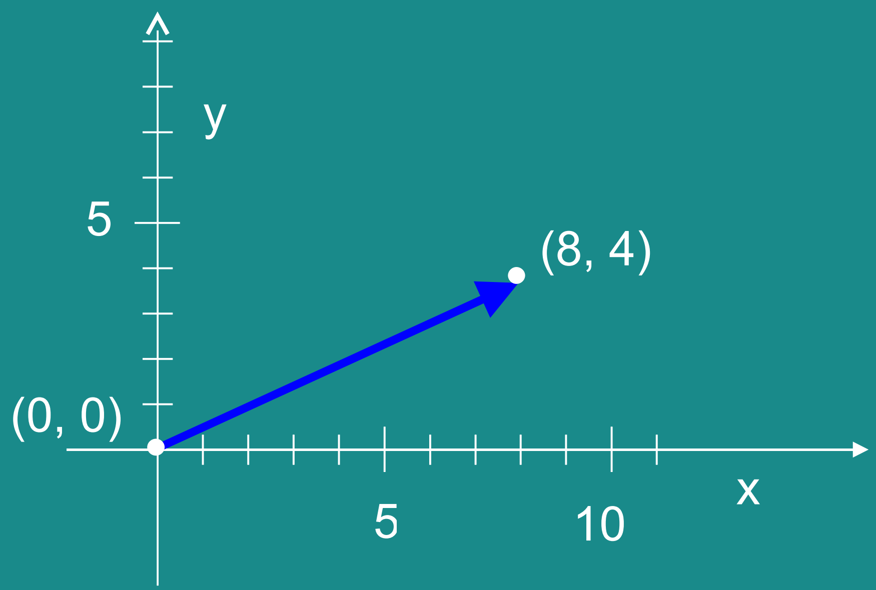


In this diagram, a force is being applied to the top left edge of the box.

IF we are concerned about the possibility of tipping over or turning, the location of the force vector IS important

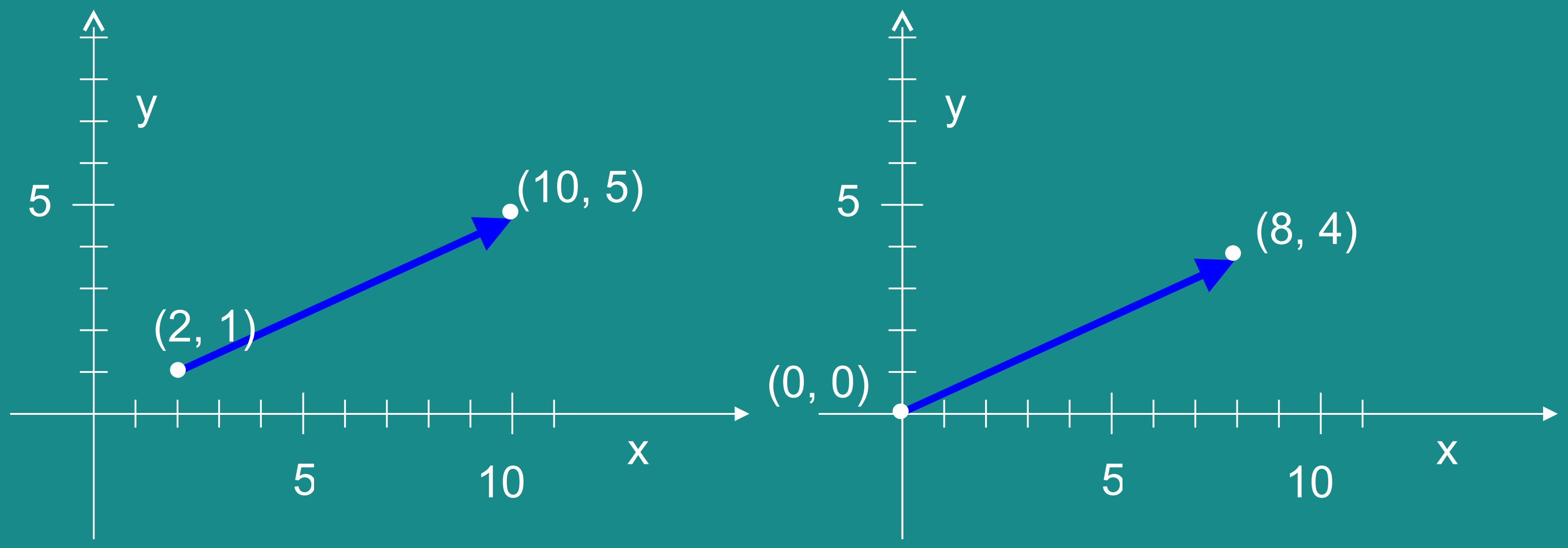
On the other hand, if we are only concerned about the box sliding to the right → where the force is applied on the left is NOT important.

- In calculus textbooks, vectors are usually drawn as starting at the origin



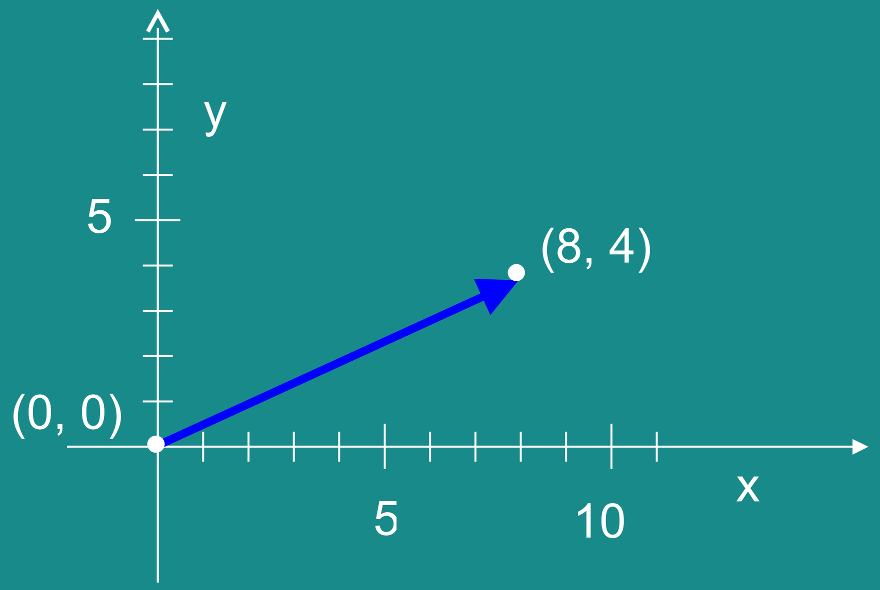
Equivalent Vectors

- These two vectors are equivalent. They have the same length and direction.



Translation

- You can move vectors around on the coordinate system. So long as you do not change their length or orientation they are equivalent. In physics we consider them to be the SAME vector.



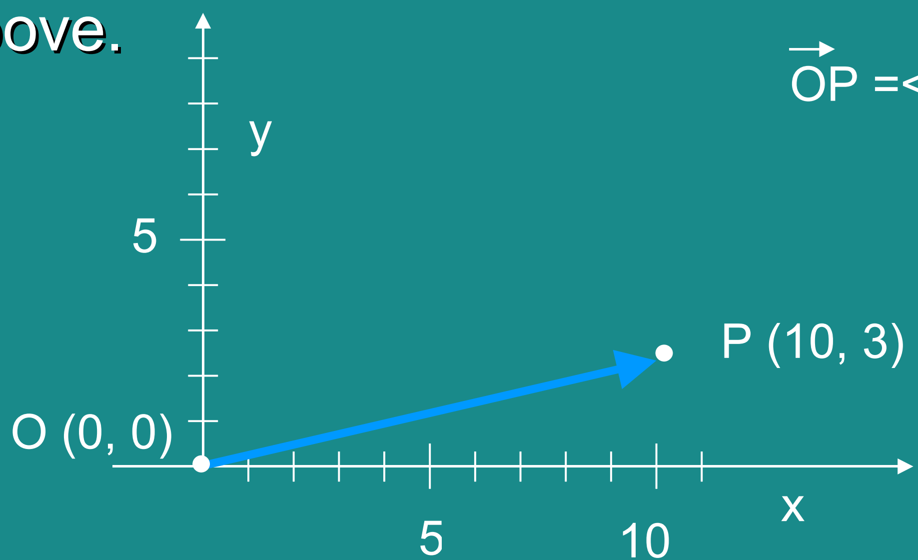
Naming Vectors

- In physics, a vector is usually named with a single letter with an arrow above it. In physics textbooks, the letter may be simply in a bold font with no arrow.

$\vec{A} = \langle 10, 3 \rangle$ $\mathbf{A} = \langle 10, 3 \rangle$

- In calculus, the letters used to describe the end points are most commonly used to name the vector, with an arrow above.

$\overrightarrow{OP} = \langle 10, 3 \rangle$

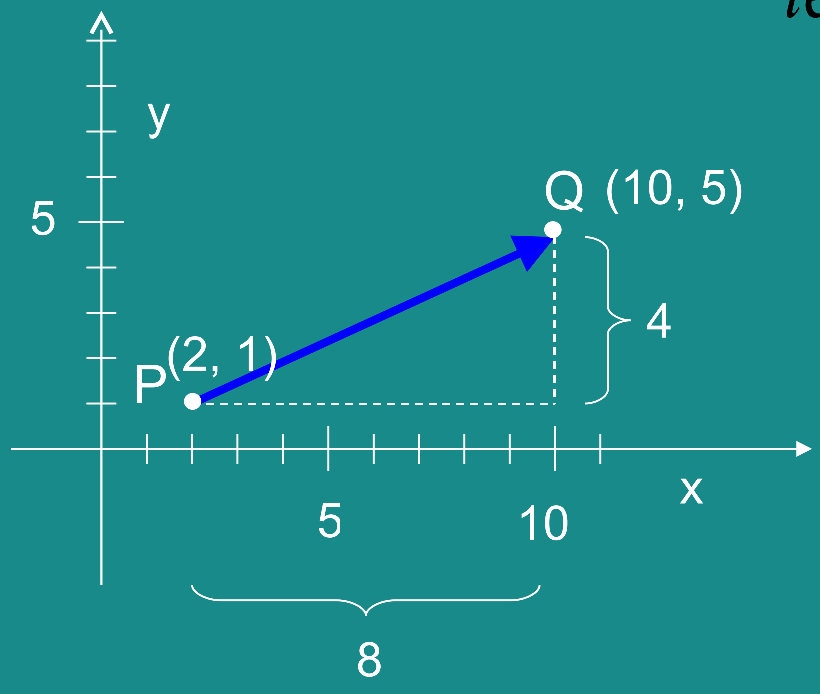


Vector Length

- To find the length (or “magnitude”) of a vector use the Pythagorean theorem.

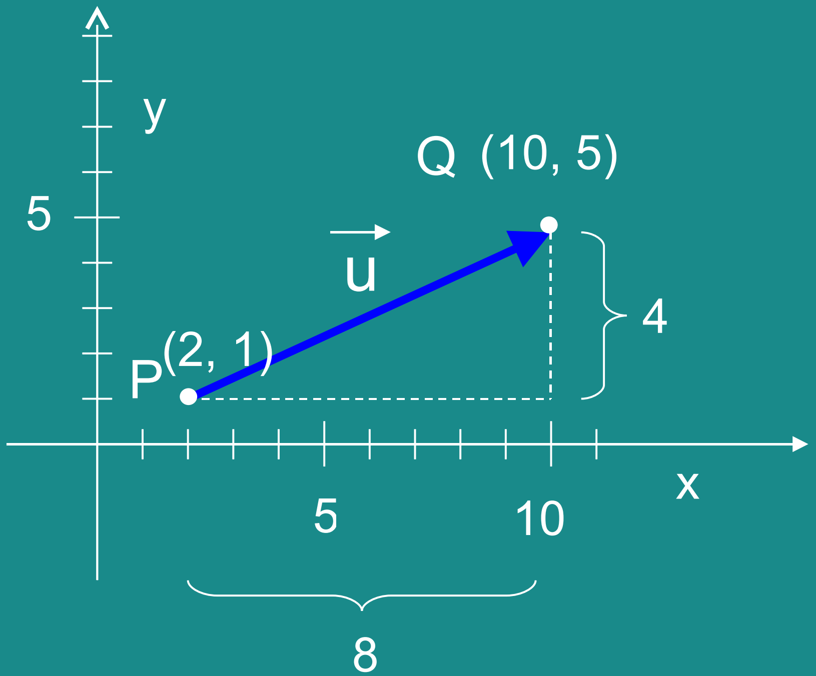
$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{8^2 + 4^2} = \sqrt{80}$$



Vector Length

Let's call the vector $\overrightarrow{PQ}$ an additional name: $\vec{u}$



$$length = \sqrt{8^2 + 4^2} = \sqrt{80} = 8.94$$

There are several different ways to denote the magnitude of the vector, for example:

$$\|\overrightarrow{PQ}\| = 8.94$$

$$|\vec{u}| = 8.94$$

$$u = 8.94$$

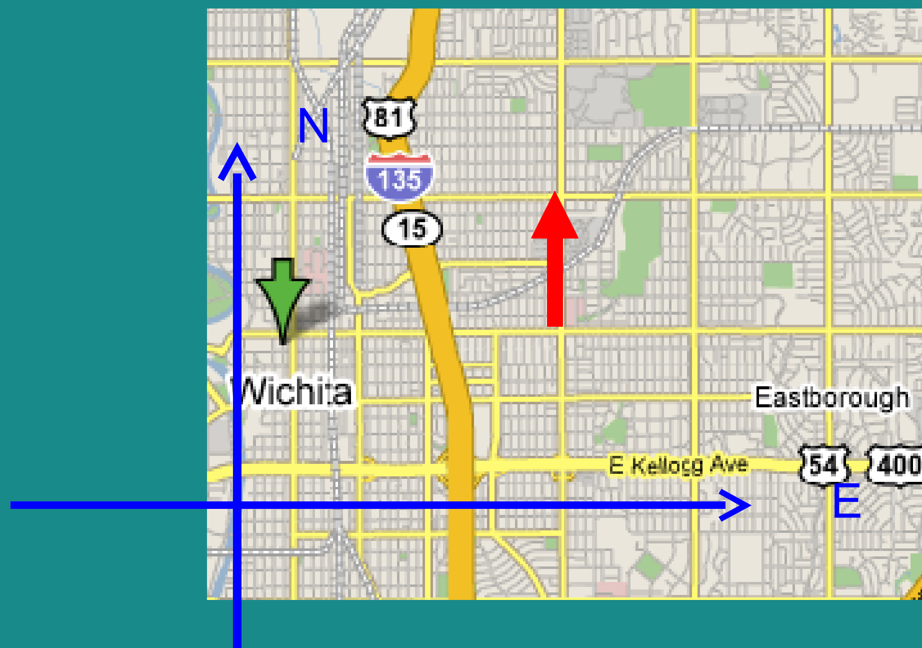
In physics texts, the symbol for the vector in regular font, with no arrow means magnitude only

Alternate method of Vector Notation

- Another, common method of expressing vectors makes use of something called “unit vectors”.
- Unit vectors code for direction, only, and have by definition a length equal to 1 unit.

- In the diagram, the red arrow indicates that some object (perhaps a car) has moved one mile north.
- Writing this as a vector, calling it the vector “d”, we could write

$\vec{d}$= 1 mile N



- If we preferred to superimpose a cartesian coordinate system on the map…

Unit vectors

- The labels N and E have been replaced with x and y.
- To indicate "in the positive y-direction" we use a unit vector
- The convention for unit vectors varies from textbook to textbook.

For the x, y and z direction we often use

$$\hat{i}, \hat{j}, \hat{k}$$

So another way to write the red vector shown is $\vec{d} = 1 \text{ mile } \hat{j}$

Other unit vector conventions

- In physics texts, the “^” symbol is used and is called “hat” so that j with the ^ above it is read as “j-hat”.
- Some physics textbooks use x, y, z instead: $\hat{y}$
- Calculus textbooks tend not to use the “^” symbol. The letters i, j, k are still used but with a vector symbol above:

For instance: $\vec{j}$

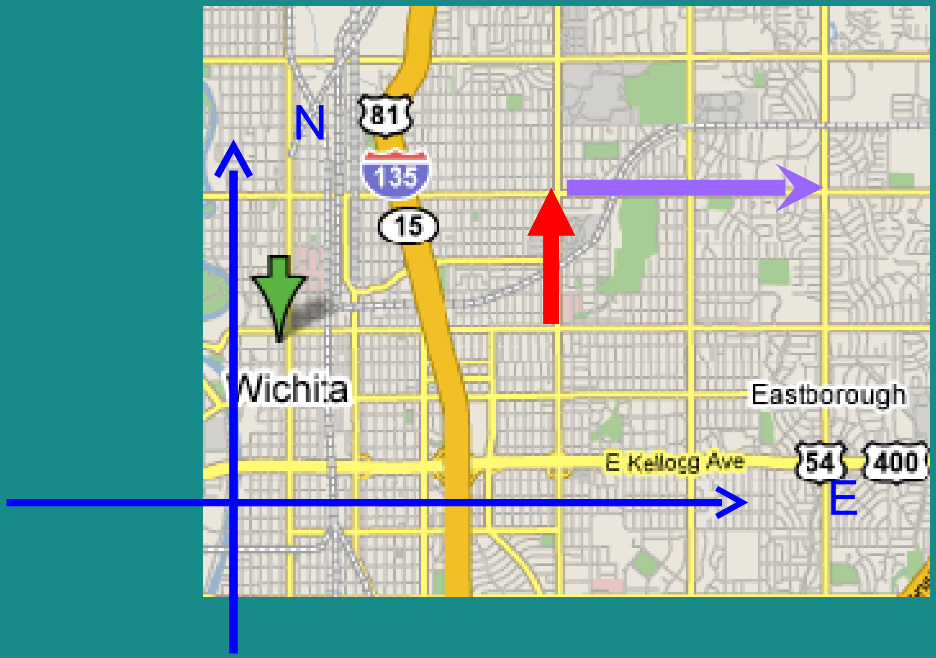
Adding Vectors Pictorially

- In the diagram, the movement of a car in two steps is indicated.

First the car heads north (the red vector) and then east (the purple vector).

These two vectors illustrate an addition.

Notice how the two vectors are arranged -the tail of the second vector is touching the tip of the first.

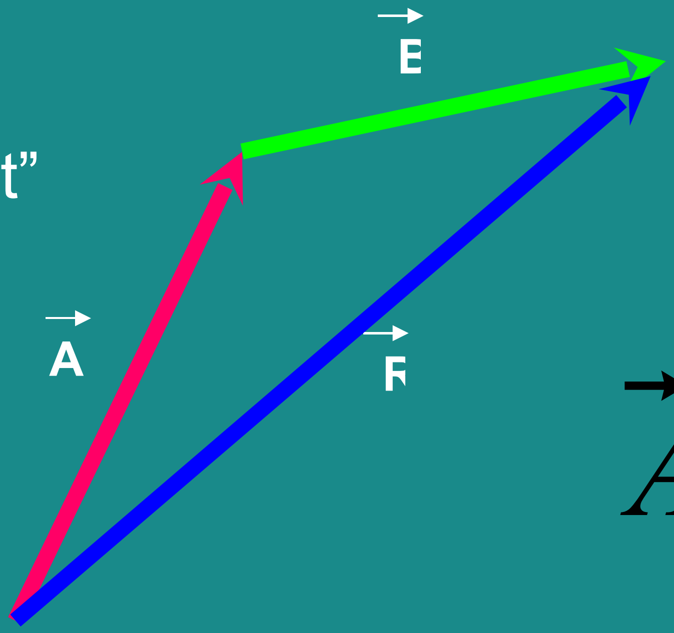


Adding vectors pictorially (cont'd)

- Here the vectors $\vec{A}$ and $\vec{B}$ are being added.

The result of the vector addition is called the "resultant"

$\vec{F}$

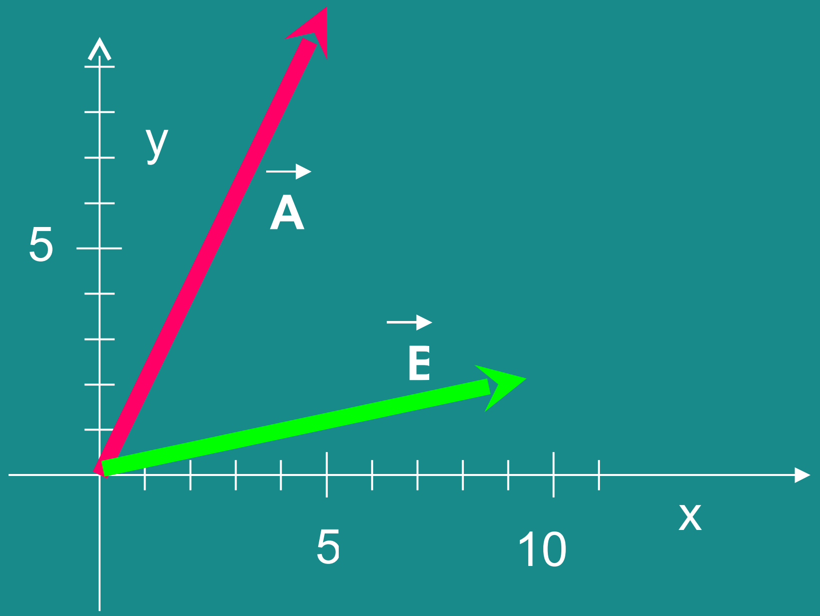


$$\vec{A}+\vec{B}=\vec{R}$$

$\vec{F}$ is the result of doing $\vec{A}$ and then $\vec{B}$.

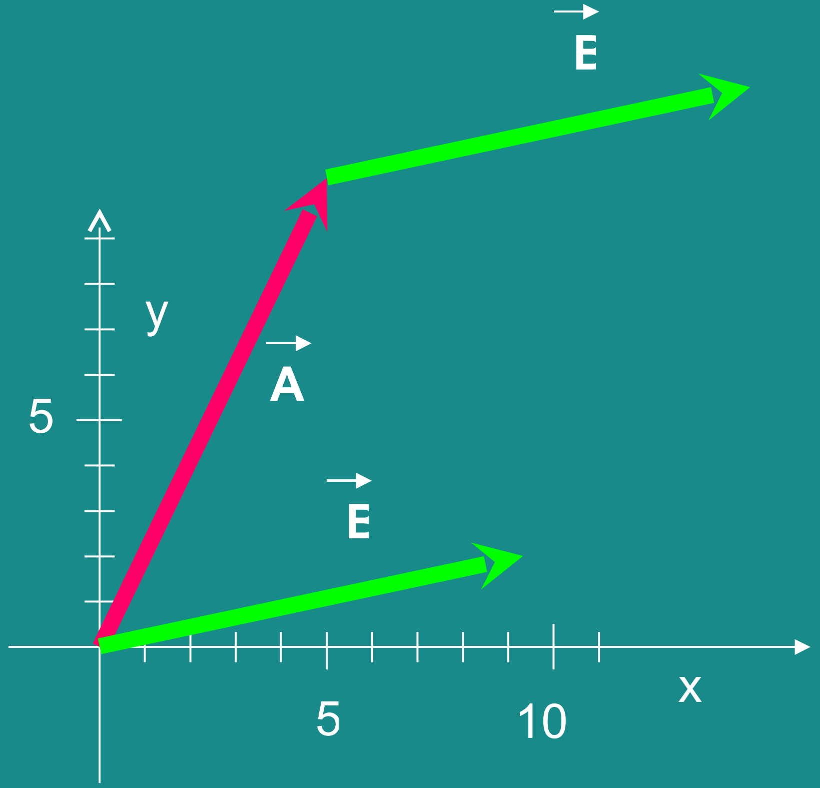
Adding pictorially

- Here are the same two vectors, drawn at the origin.
- In order to add the vectors, we must move one of them so that it's base is at the tip of the other.



Adding pictorially

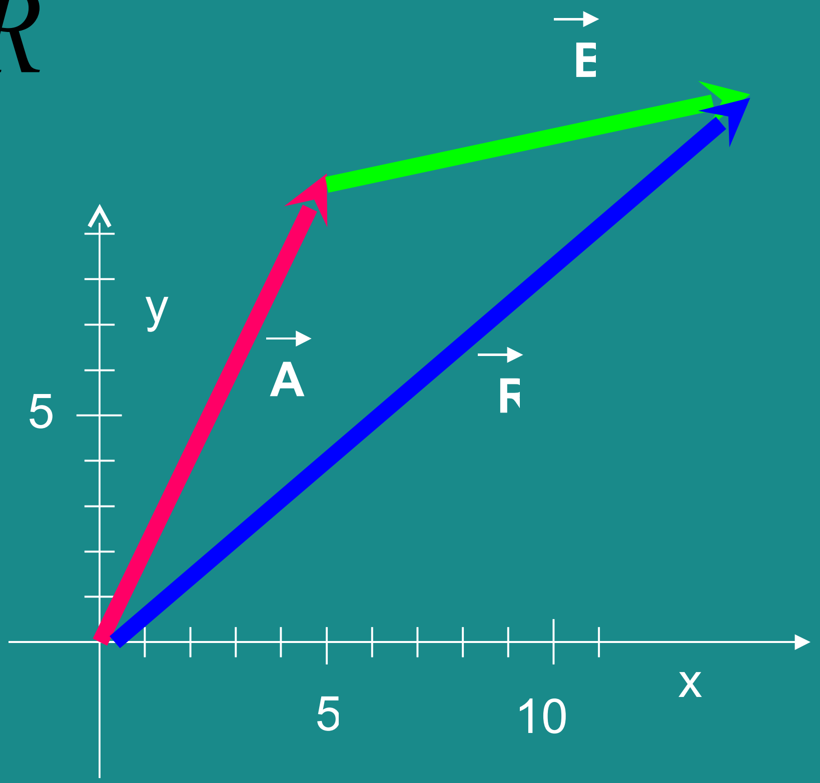
- Remember, translating a vector does not change it.



Adding pictorially

- Now the resultant is drawn

$$\vec{A} + \vec{B} = \vec{R}$$



Components using unit vectors

- A simpler way to add vectors involves simply adding their components

$\vec{B} = \langle 1, -4 \rangle$

$\vec{A} = \langle 10, 3 \rangle$

Then to find $\vec{A} + \vec{B} = \vec{R}$

Add the x-components of the two vectors and the y components of the two vectors (but don't mix x and y)

$\vec{R} = \langle 10+1, 3+(-4) \rangle = \langle 11, -1 \rangle$

Adding by components

- In other words, if the resultant vector R can be represented by $<R_x, R_y, R_z>$ then the addition of two vectors A and B which are represented by $<A_x, A_y, A_z>$ and $<B_x, B_y, B_z>$ then the vector R is given by

$$R = <R_x, R_y, R_z> = <A_x + B_x, A_y + B_y, A_z + B_z>$$

Finding components via Trig

- Often in physics, the components of a vector are not given.
- The components need to be determined by the student from the angle and the magnitude of the vector.
- Usually a 2D problem, not 3D
- This is accomplished using basic trigonometry.

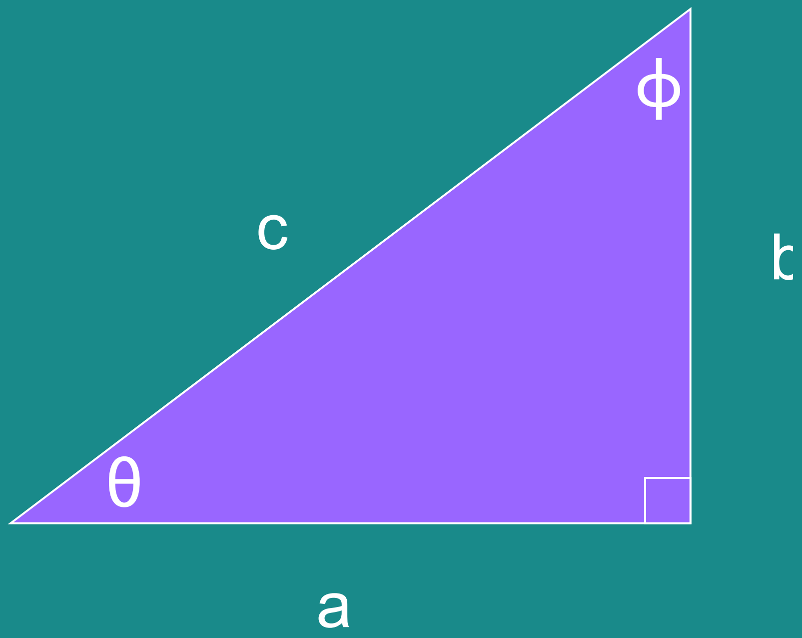
Trigonometric Definitions

- You remember these basic definitions:

$$\sin\theta = \frac{b}{c}$$

$$\cos\theta = \frac{a}{c}$$

$$\tan\theta = \frac{b}{a}$$

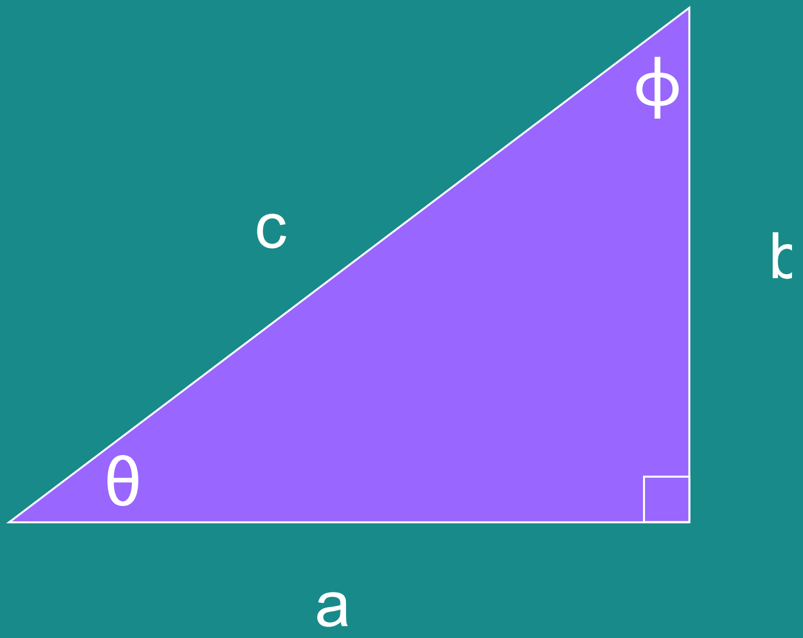


Trigonometric Definitions

- These can be rearranged to solve for the base and vertical side of the right angle triangle in terms of the angle and magnitude of the hypotenuse:

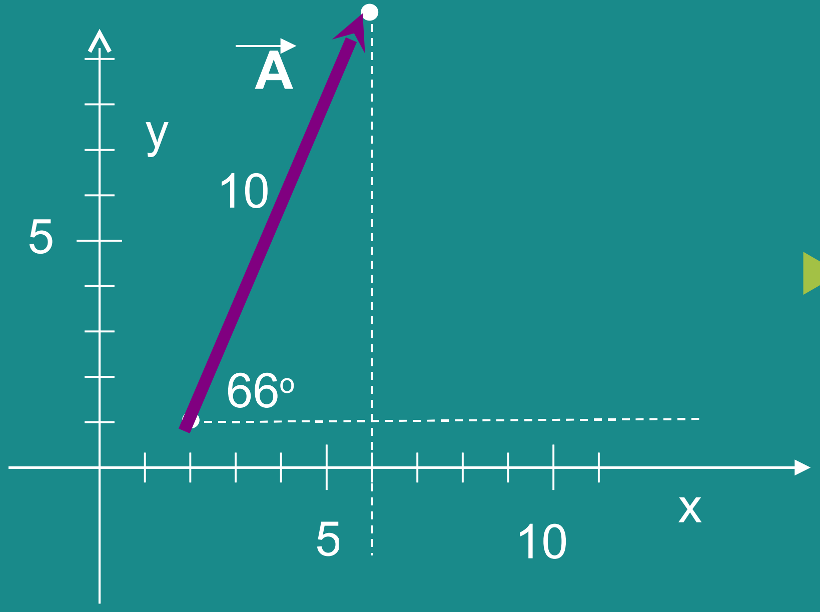
$$b = c \sin \theta$$

$$a = c \cos \theta$$



Writing a vector in components

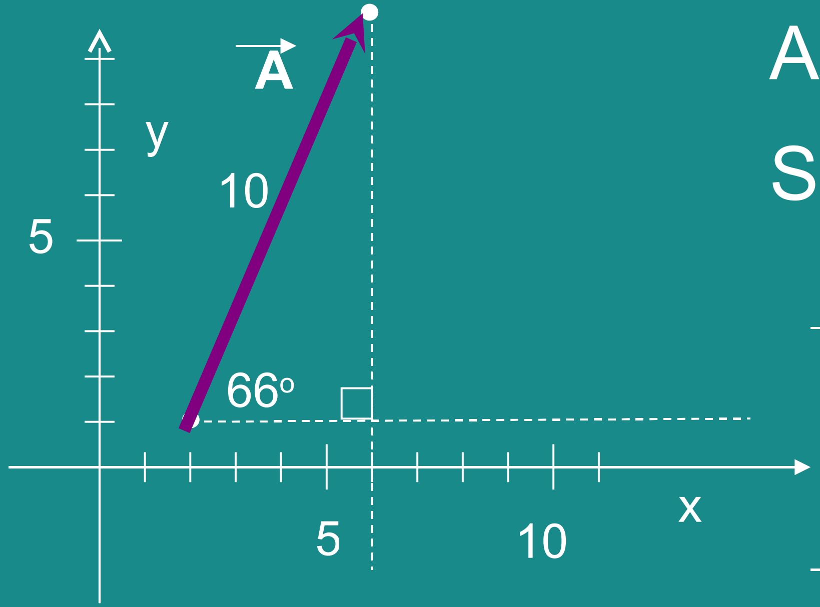
- Consider the vector shown which we know has a length (magnitude) of 10 and makes an angle of 66° with the x-direction.



- The vector forms the hypotenuse, and we need to draw in the other two sides of the triangle.
- Drop a vertical line from the tip of the arrow

Writing a vector in components

- The horizontal and vertical sides of the triangle are the x and y components of the vector
- These are calculated using the trig definitions



$A_x = 10 \cos(66^o) = 4.1$

$A_y = 10 \sin(66^o) = 9.1$

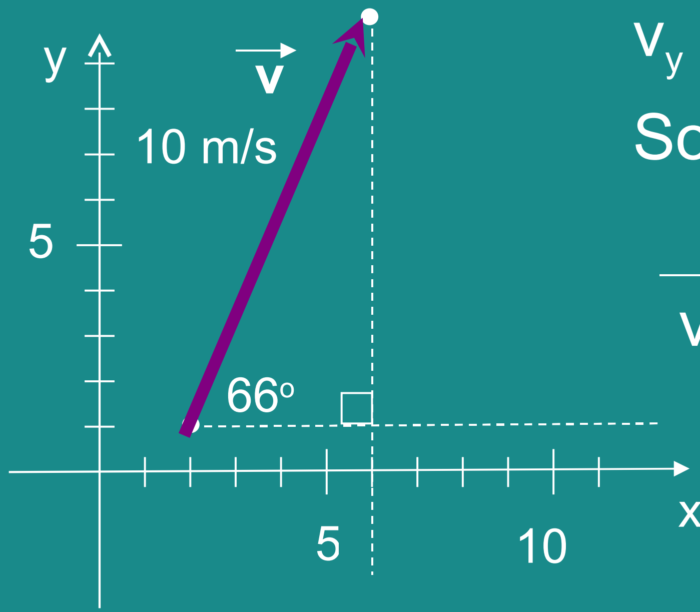
So the vector can be written:

$\vec{A} = <4.1, 9.1>$ or as

$\vec{A} = 4.1\,\hat{i} + 9.1\hat{j}$

More on components

- If the vector represents a physical quantity – such as a velocity or force – the vector will have units of measurement
- Let's take the same vector and write it as a velocity vector, which has units m/s.
- The units just "tag along".



$v_x = 10 \text{ m/s} \cos(66^o) = 4.1 \text{ m/s}$

$v_y = 10 \text{ m/s} \sin(66^o) = 9.1 \text{ m/s}$

So the vector can be written:

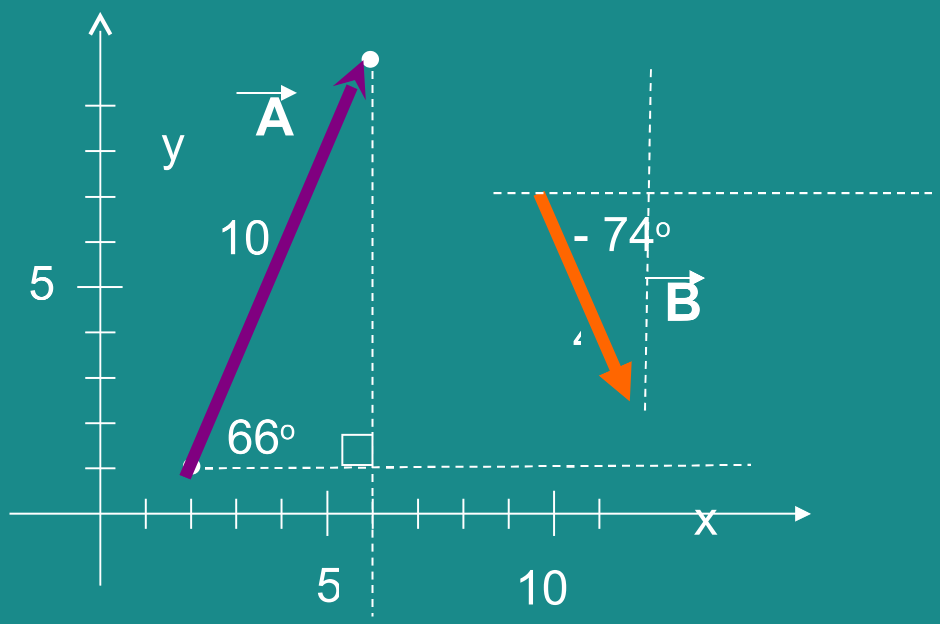
$\vec{v} = \langle 4.1 \text{ m/s}, 9.1 \text{ m/s} \rangle$ or as

$\vec{v} = 4.1 \text{ m/s}\ \hat{i} + 9.1 \text{ m/s}\ \hat{j}$

Adding 2D vectors using components

- If two vectors are being added, first write the vectors in terms of their components:

$\vec{A} = \langle 4.1, 9.1 \rangle$ or $\vec{A} = 4.1\,\hat{i} + 9.1\hat{j}$



$B_x = 4 \cos(-74^o) = 1.1$

$B_y = 4 \sin(-74^o) = -3.8$

So that

$\vec{B} = \langle 1.1, -3.8 \rangle$ or

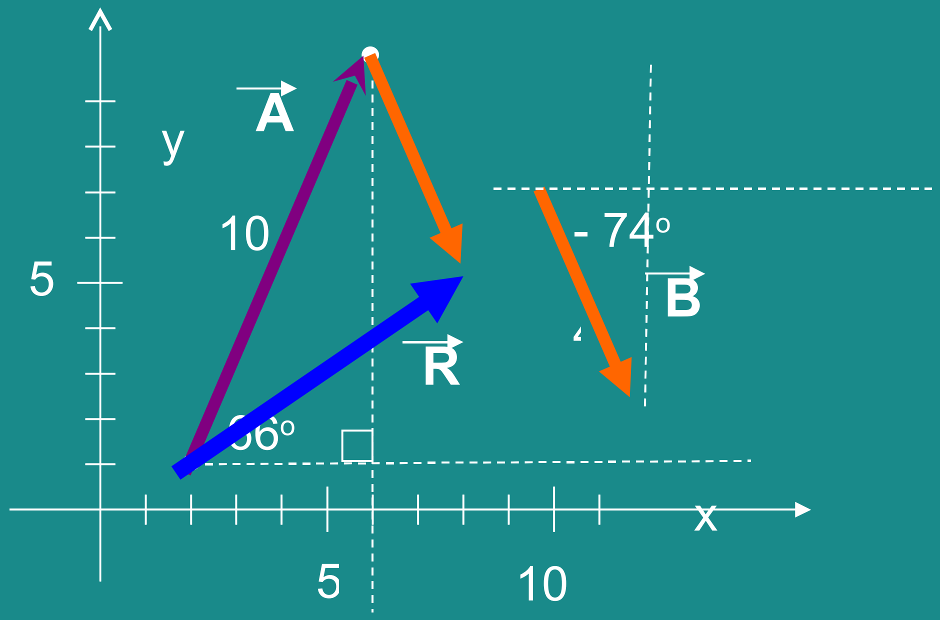
$\vec{B} = 1.1\,\hat{i} - 3.8\,\hat{j}$

Notice the y-component of the orange vector is negative – it points downwards!

- Then, as before, add the components to find the resultant.

$$\vec{R} = \langle R_x, R_y \rangle = \langle A_x + B_x, A_y + B_y \rangle$$

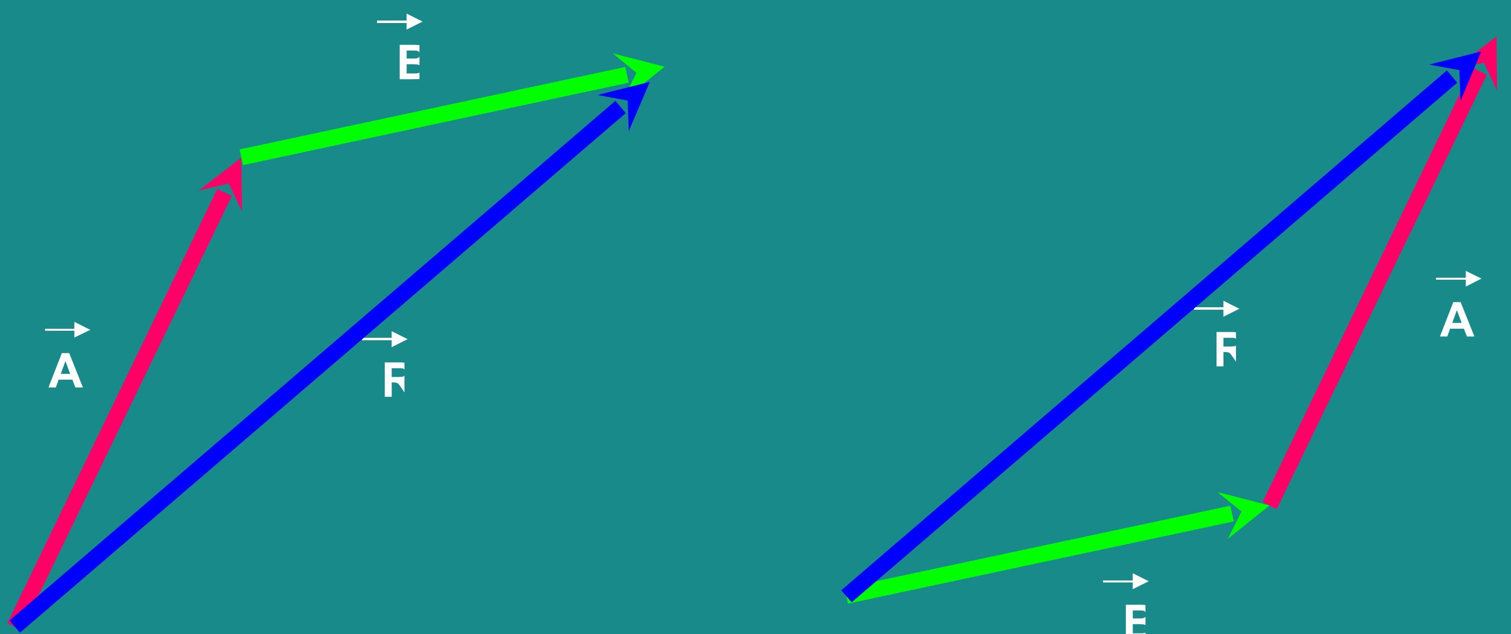
$$\vec{R} = \langle R_x, R_y \rangle = \langle 4.1 + 1.1, 9.1 + -3.8 \rangle = \langle 5.2, 5.3 \rangle$$



Order of addition

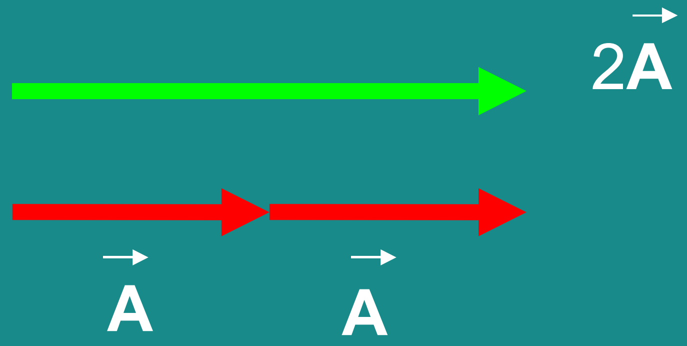
- The order in which you add vectors does not matter.

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$



Multiplication by a Scalar

- What do you suppose the meaning of $2\vec{A}$ is?
- The green vector is twice as long as the red vector.
- (click to see this) if you add the vector $\vec{A}$ to itself, you get a vector equivalent to the green vector.

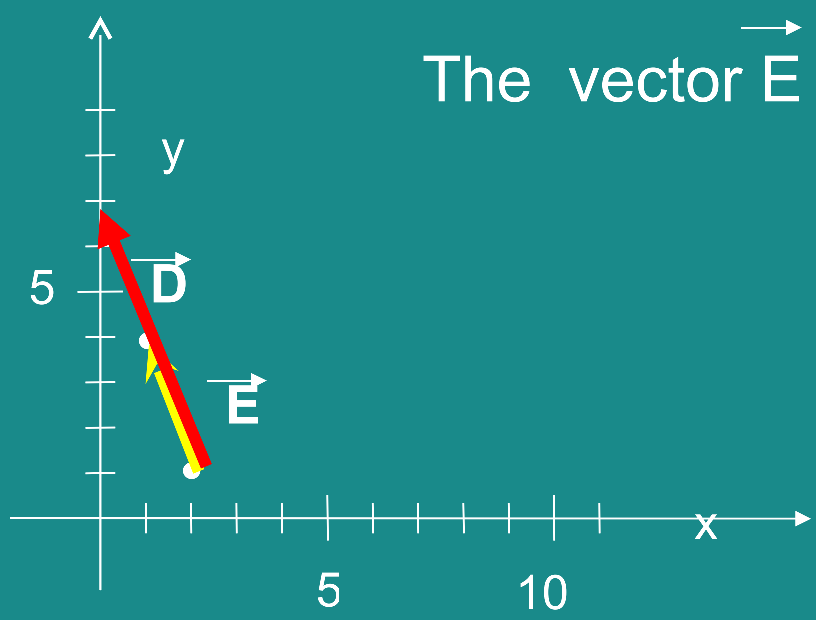


So $\vec{A} + \vec{A} = 2\,\vec{A}$

Multiplication by Scalar (cont'd)

- So it is apparent that multiplying by a number greater than one increases the magnitude of a vector.
- Multiplying by a number smaller than one, shrinks it
- This can be seen by working with the components as well…

Take the vector $\vec{D} = <-2, 6>$ shown in red



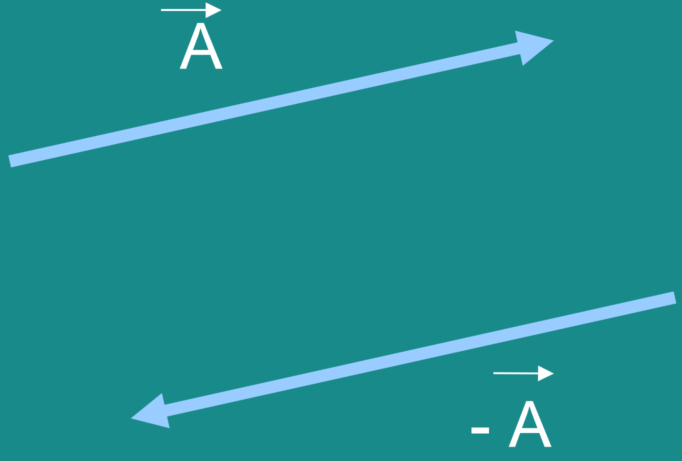
The vector $\vec{E} = 1/2\ \vec{D}$

$= 1/2 <-2, 6>$

$= <-1, 3>$

Scalar multiplication

- Multiplying by a negative number reverses the direction of the vector:

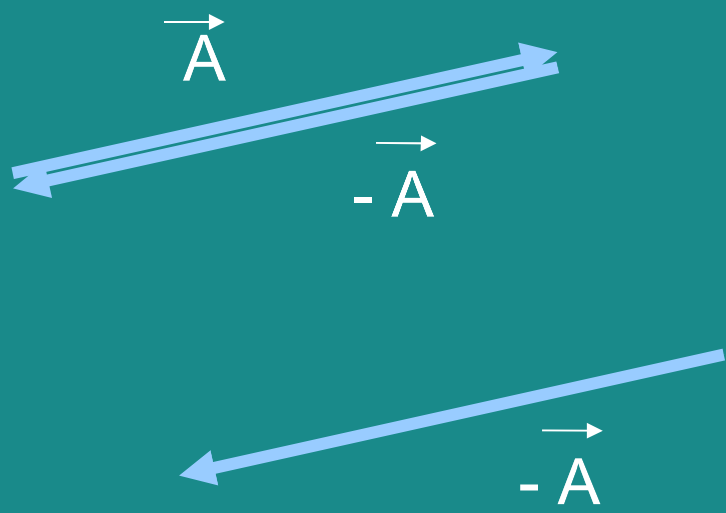


Vector Subtraction

- In algebra, subtraction is sometimes thought of as "adding a negative"
- The same idea works with vectors.

What should $\vec{A} – \vec{A}$ equal? Zero!

Using the visual arrows $\vec{A} +(-\vec{A})$ ends up back at the starting point.



Dot Product (vector multiplication)

- The dot product (also called scalar product because the result is a scalar) is formed between two vectors
- The "dot" looks like the symbol used in regular multplication 6 = 2·3, but means something different. A vector is not a single number.

$$\vec{A} \cdot \vec{B}$$

Dot product

- The dot product can be calculated using either

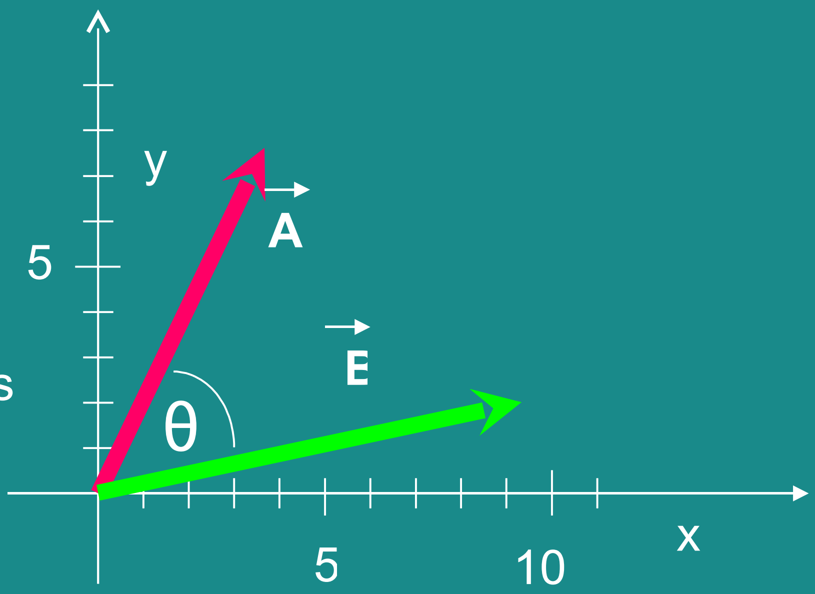
$$\vec{A}\cdot\vec{B} = < A_x, A_y, A_z > ¿ < B_x, B_y, B_z > ¿$$

$$¿ \; = A_x B_x + A_y B_y + A_z B_z$$

or

$$\vec{A}\cdot\vec{B} = AB\cos(\theta)$$

Where θ is the angle between the vectors



Dot product

- In physics, the angle and the magnitude are usually known.
- In calculus, the equation is usually rearranged so that the angle can be determined from the dot product.

$$\vec{A} \cdot \vec{B} = AB\cos(\theta)$$

Physics version

$$\cos(\theta) = \frac{\vec{A} \cdot \vec{B}}{AB}$$

Calculus version